APPROXIMATE SOLUTION OF A SECOND-ORDER LINEAR DIFFERENTIAL EQUATION WITH VARIABLE COEFFICIENTS

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The equations in this paper are solved by a method of successive approximations which is an extension of earlier work [1]. The convergence of the successive approximations is proved and it is shown that the original approximation gives satisfactory results for a certain class of problems.

1. Statement of the problem. We consider the equation

$$\alpha y'' + f(x)\, y' + F(x)\, y = 0 \qquad \left(y' = \frac{dy}{dx},\ y'' = \frac{d^2y}{dx^2},\ \alpha = \text{const}\right) \tag{1.1}$$

Here, x and y are real variables; $f(x)$ and $F(x)$ are real, continuous functions with continuous derivatives $f'(x)$, $F'(x)$, and $F(x) \neq 0$.

We shall seek the general solution of Equation (1.1) with the aid of relations [1]

$$\psi = A_1 e^{r_1 x} + A_2 e^{r_2 x}, \qquad y' = A_1 r_1 e^{r_1 x} + A_2 r_2 e^{r_2 x}, \qquad y'' = A_1 r_1^2 e^{r_1 x} + A_2 r_2^2 e^{r_2 x} \tag{1.2}$$

Here r_1 and r_2, the roots of the characteristic equation

$$r^2 + mr + p = 0 \tag{1.3}$$

are given by

$$\begin{aligned} r_1 &= -\frac{m}{2} + k, \quad r_2 = -\frac{m}{2} - k, \quad \text{if} \quad k^2 = \left(\frac{m^2}{4} - p\right) > 0 \qquad (r_1 \neq r_2) \\ r_1 &= -\frac{m}{2} + qi, \quad r_2 = -\frac{m}{2} - qi, \quad \text{if} \quad q^2 = \left(p - \frac{m^2}{4}\right) > 0 \qquad (i = \sqrt{-1}) \end{aligned} \tag{1.4}$$

The parameters m and p are to be determined later; the function ψ is defined by the relation

$$\psi = \left[\frac{f(x) - m}{p}\right] y' + \frac{F(x)}{p} y \tag{1.5}$$

In accordance with (1.2) the functions A_1 and A_2 have to be determined from the system of differential equations

$$A_1' e^{r_1 x} + A_2' e^{r_2 x} = \mu, \qquad A_1' r_1 e^{r_1 x} + A_2' r_2 e^{r_2 x} = 0$$
$$\mu = \psi' - y', \qquad A_1' = \frac{dA_1}{dx}, \qquad A_2' = \frac{dA_2}{dx}, \qquad \psi' = \frac{d\psi}{dx} \tag{1.6}$$

Substituting ψ from (1.5) into the first of the equations (1.2), and taking into consideration the second one, we obtain

$$y = \frac{1}{F(x)} \left\{ [p - (f(x) - m) r_1] A_1 e^{r_1 x} + [p - (f(x) - m) r_2] A_2 e^{r_2 x} \right\} \tag{1.7}$$

On the basis of Equations (1.6), (1.1) and (1.5) the expression for μ can be presented in the form

$$\mu = \left[\frac{f(x) - m}{p} - \frac{\alpha F'(x)}{pF(x)}\right] y'' + \left[\frac{f'(x) + F(x) - p}{p} - \frac{F'(x) f(x)}{pF(x)}\right] y' \tag{1.8}$$

From the differential equations (1.6) we obtain

$$A_1 = A_{10} - \frac{r_2}{(r_1 - r_2)} \int_{x_0}^{x} \mu e^{-r_1 x} dx, \qquad A_2 = A_{20} + \frac{r_1}{(r_1 - r_2)} \int_{x_0}^{x} \mu e^{-r_2 x} dx \tag{1.9}$$

Here A_{10} and A_{20} are constants of integration; they are the values of the functions A_1 and A_2 at the initial instant $x = x_0$, $y = y_0$, $y' = y_0' = z_0$.

The values of A_{10} and A_{20} are found if one sets $x = x_0$ in the first equation of (1.2) and solves the obtained system

$$A_{10} = \left(\frac{z_0 - \psi_0 r_2}{r_1 - r_2}\right) e^{-r_1 x_0}, \qquad A_{20} = \left(\frac{\psi_0 r_1 - z_0}{r_1 - r_2}\right) e^{-r_2 x_0}, \qquad \psi_0 = \frac{[f(x_0) - m] z_0 + F(x_0) y_0}{p} \tag{1.10}$$

If the given equation (1.1) has constant coefficients $f(x) \equiv$ const, $F(x) \equiv$ const, then one can let $f(x) \equiv m$, and $F(x) \equiv p$, and by (1.8) the function μ will be identically zero, while the functions A_1 and A_2, in view of (1.6), will be simply constants of integration

$$A_1 \equiv A_{10} = \text{const}, \quad A_2 = A_{20} = \text{const}$$

In the general case, when the coefficients of the given equation (1.1) are variables, the function μ will not vanish identically, and the functions A_1 and A_2 will not be constants. One has to determine them. We shall show that the functions A_1 and A_2 can be found by the method of successive approximations.

2. Let us return to the consideration of the differential equation (1.6) or Equation (1.9). The function μ is determined by means of (1.8), that is, it depends on the functions $f(x)$ and $F(x)$, and also on y' and y''. Let us express the function μ in terms of the variables A_1, A_2, and x. If we substitute y'' and y' from (1.2) into (1.8), and make use of (1.6), we obtain the system of linear homogeneous differential equations

$$A_1' = \theta_1(x) A_1 + \theta_2(x) A_2, \qquad A_2' = \theta_3(x) A_1 + \theta_4(x) A_2 \tag{2.1}$$

The coefficients $\theta_1(x)$, ..., $\theta_4(x)$ are functions depending on the functions $f(x)$, $f'(x)$, $F(x)$, $F'(x)$, and on the parameters m and p. But in any case, the coefficients of Equation (2.1) will be continuous functions of the variable x. Under these circumstances it will be possible to determine the functions A_1 and A_2 by the method of successive approximations [2]. The convergence of the successive approximations is ensured in the interval of continuity of the functions $f(x)$, $F(x)$, $f'(x)$ and $F'(x)$. The rate of convergence depends on the type of the latter functions and on the values of the parameters m and p. In the sequel we shall find various approximations for the functions A_1 and A_2 making use of the integral relations (1.9). We add that in some cases the functions $f(x)$ and $F(x)$ can be of such a type that the performance of the integration in the finding of the successive approximations may be quite difficult from a practical viewpoint. One can circumvent this difficulty if one uses for y not the expression (1.7) but a relation obtained by integrating the second equation in (1.2). We have

$$y = y_0 + \int_{x_0}^{x} A_1 r_1 e^{r_1 x} dx + \int_{x_0}^{x} A_2 r_2 e^{r_2 x} dx \tag{2.2}$$

When the functions A_1 and A_2 are known by some approximation, we find μ with the aid of (2.2) and (1.2). After this we find, by the use of (1.9), the expressions A_1 and A_2 in the next approximation; by substituting the refined expressions for A_1 and A_2 into (1.2), we find the refined general solution.

Depending on the degree of accuracy of A_1 and A_2, one finds various approximations for the general solution of Equation (1.1). One obtains the general solution as the starting and simplest (the zeroth) approximation if one sets $A_1 \equiv A_{10} = \text{const}$, $A_2 \equiv A_{20} = \text{const}$, that is, if one uses in place of the variable functions A_1 and A_2 only their initial values. By (1.2) we have in this case

$$y = y(x), \qquad y' = y'(x), \qquad y'' = y''(x)$$

In order to obtain the functions A_1 and A_2 in the next, the first approximation, we substitute into (1.8) the expressions y' and y'' from the zeroth approximation. We find the expression for the function μ in terms of x, and the constants A_{10} and A_{20}. Carrying out the integration in accordance with (1.9), we find the first approximation of the functions $A_1 = A_1^{(1)}$, $A_2 = A_2^{(1)}$. Substituting into the formulas (1.2) in place of A_1 and A_2 the functions $A_1^{(1)}$ and $A_2^{(1)}$ we find the first approximation of the general solution of the equation. On the basis of the first approximation we now find the second approximation, and so on.

The presented process of successive approximations shows why the given method of solving the equation will yield a greater accuracy in computations than the usual method of successive approximations. In the method presented the zeroth approximation corresponds to the case when A_1 and A_2 are constants. The variable y in the zeroth approximation is obtained on the basis of a general solution that depends on x in a definite manner; furthermore, for linear equations with constant coefficients, the zeroth approximation coincides with the exact solution. In the usual method of successive approximations the zeroth solution consists simply of the initial value of the dependent variable and its derivatives, $y = y_0 =$ const, $y' = y_0' = z_0 =$ const.

3. In order to obtain the zeroth approximation of the general solution of Equation (1.1) one must let $A_1 \equiv A_{10} =$ const, $A_2 \equiv A_{20} =$ const, and replace the functions A_1 and A_2 by A_{10} and A_{20} in (1.2). Thus we obtain

$$\left[\frac{f(x) - m}{p}\right] y' + \frac{F(x)}{p} y = A_{10} e^{r_1 x} + A_{20} e^{r_2 x} \tag{3.1}$$

$$y' = A_{10} r_1 e^{r_1 x} + A_{20} r_2 e^{r_2 x} \tag{3.2}$$

$$y'' = A_{10} r_1^2 e^{r_1 x} + A_{20} r_2^2 e^{r_2 x} \tag{3.3}$$

The constants of integration A_{10} and A_{20} are found by means of the conditions $x = x_0$, $y = y_0$, $y' = y_0' = z$ from (1.10). Hence, the zeroth approximation of the solution is obtained without solving a system of auxiliary differential equations, and it can be written down for arbitrary functions $F(x)$ and $F(x)$ regardless of the manner in which these functions are given.

If one eliminates y' from (3.1) and (3.2) and replaces A_{10} and A_{20} by their expressions from (1.10), then one obtains a relation between y and x as the zeroth approximation for arbitrary functions $f(x)$ and $F(x)$ and the real parameter k:

$$y = \frac{1}{F(x)} e^{-1/2 m (x - x_0)} \left[(ab - cd) \cosh k (x - x_0) + \left(\frac{bd - ack^2}{k}\right) \sinh k (x - x_0)\right] \tag{3.4}$$

If k is imaginary (q real) the corresponding expression is

$$y = \frac{1}{F(x)} e^{-\frac{1}{2}m(x-x_0)} \left[(ab - cd) \cos q(x - x_0) + \left(\frac{bd + acq^2}{q} \right) \sin q(x - x_0) \right] \quad (3.5)$$

In the last formulas

$$a = \frac{F(x_0)\, y_0 + [f(x_0) - m]\, z_0}{p}, \qquad b = p + \frac{1}{2} m\, [f(x) - m]$$

$$c = f(x) - m, \quad d = z_0 + \frac{1}{2} am \quad (3.6)$$

4. The general solution of Equation (1.1), found by the presented method, will contain parameters m and p which are still undetermined. Furthermore, in accordance with (1.4) it will also involve undetermined roots r_1 and r_2. For linear equations with constant coefficients, $f(x) \equiv$ const, $F(x) \equiv$ const, and if one chooses the parameters so that $m \equiv f(x) =$ const, $f'(x) \equiv 0$, $p \equiv F(x) =$ const, $F'(x) = 0$, then in accordance with (1.8) the function $\mu \equiv 0$, and one finds from (1.6) that $A_1' \equiv A_2' = 0$, that is, one arrives at the known exact solution. For an equation with variable coefficients, the parameters m and p can be chosen in various ways depending on the type of the considered equation (1.1) and the statement of the problem (the searching for a solution of the equation over a short interval of x, the investigation of the solution for large values of x, and others). One can, however, make some statement in regard to the choice of the parameters m and p if one is guided by the idea of determining the best approximation to the exact solution and of keeping the computation as simple as possible. The greatest simplicity in the computations by the present method is obtained when one uses the zeroth approximation, i.e. when one sets $A_1 \equiv A_{10}$, $A_2 \equiv A_{20}$. Let us write down the system (1.6) for the given case assuming that we know the initial conditions $x = x_0$, $y = y_0$, $y' = y_0' = z$, and bearing in mind that $A_1' = A_2' = 0$. We find the unique relation $\mu = 0$, which according to (1.8) takes the form

$$\left[\frac{f(x_0) - m}{p} - \frac{\alpha F'(x_0)}{pF(x_0)} \right] y_0'' + \left[\frac{f'(x_0) + F(x_0) - p}{p} - \frac{F'(x_0) f(x_0)}{pF(x_0)} \right] y_0' = 0 \quad (4.1)$$

Depending on the type of the functions $f(x)$ and $F(x)$ in (1.1), the variables y' and y'' can have most various forms, but in every case $y_0' \not\equiv 0$, $y'' \not\equiv 0$. Therefore, Equation (4.1) will be satisfied if one equates to zero the coefficients of y_0' and y_0''. Imposing these conditions, we obtain relations between the parameter m, p and the values of the given functions $f(x_0)$ and $F(x_0)$ at the initial point

$$m = m_0 = f(x_0) - \alpha \frac{F'(x_0)}{F(x_0)} \quad (4.2)$$

$$p = p_0 = F(x_0) + f'(x_0) - \frac{F'(x_0) f(x_0)}{F(x_0)} \quad (4.3)$$

It can be seen from the last formula that for a linear equation with constant coefficients $f(x) \equiv \text{const}$, $f'(x) \equiv 0$, $F(x) \equiv \text{const}$, $F'(x) \equiv 0$, the parameters m and p will also be constants for all values of x. For equations with variable coefficients, the parameters m and p, determined by (4.2) and (4.3), will depend on the argument x_0. This means that in attempting to solve Equation (1.1) by the presented method one must choose m and p in the zeroth approximation in various ways depending on the argument x_0; the zeroth approximation will be usable (for a given choice of x_0) over an interval $(x_1 - x_0)$ of x if m and p do not change too much.

Subsequently one must compute new values of the parameters by means of (4.2) and (4.3) for the new initial value of the argument x_1, and apply anew the formulas of the zeroth approximation, and so on. Instead of selecting various values m and p for particular intervals of the variable x (m_0, p_0 for $x_0 < x \leqslant x_1$, m_1, p_1 for $x_1 < x < x_2$, m_n, p_n for $x_n < x \leqslant x^\circ$) one can select the mean values of the parameters in the considered interval $x_0 < x \leqslant x^\circ$, that is, one can compute them by means of the formulas

$$m^\circ = \frac{1}{x^\circ - x_0} \int_{x_0}^{x^\circ} m\,dx, \qquad p^\circ = \frac{1}{x^\circ - x_0} \int_{x_0}^{x^\circ} p\,dx$$

Since the end, $x = x^\circ$, of the interval under consideration is not fixed in any way, we shall write in the sequel in place of x° simply x, and in place of m° and p°, simply m and p.

$$m = \frac{1}{x - x_0} \int_{x_0}^{x} \left[f(x) - \frac{\alpha F'(x)}{F(x)} \right] dx = \frac{1}{x - x_0} \int_{x_0}^{x} f(x)\,dx - \frac{\alpha}{x - x_0} \ln \frac{F(x)}{F(x_0)} \tag{4.4}$$

$$p = \frac{1}{x - x_0} \int_{x_0}^{x} \left[F(x) + f'(x) - \frac{F'(x) f(x)}{F(x)} \right] dx = \frac{f(x) - f(x_0)}{x - x_0} + \frac{1}{x - x_0} \int_{x_0}^{x} \left[F(x) - \frac{F'(x) f(x)}{F(x)} \right] dx \tag{4.5}$$

From this, it is necessary to ensure that the roots r_1 and r_2 are distinct. In the opposite case, the parameters m and p have to be determined in a different way. In particular, one can average the functions $f(x)$ and $F(x)$ directly, by setting

$$m = \frac{1}{x - x_0} \int_{x_0}^{x} f(x)\,dx, \qquad p = \frac{1}{x - x_0} \int_{x_0}^{x} F(x)\,dx$$

5. As an illustration and a simultaneous appraisal of the method presented let us consider Bessel's equation

$$y'' + \frac{1}{x} y' + x = 0 \qquad (x > 0) \tag{5.1}$$

We apply the presented method. According to (1.1) and (5.1) we have

$$\alpha \equiv 1, \qquad f(x) = \frac{1}{x}, \qquad f'(x) = -\frac{1}{x^2}, \qquad F(x) \equiv 1, \qquad F'(x) \equiv 0$$

The characteristic equation (1.3) for this case takes on the form

$$r^2 + mr + p = 0, \qquad r_1 = -\frac{1}{2} m + k, \qquad r_2 = -\frac{1}{2} m - k, \qquad k = \sqrt{\frac{1}{4} m^2 - p} \tag{5.2}$$

By (1.7) and (1.2) we have the general solution of Equation (5.1) in the form

$$y = \left[p - \left(\frac{1}{x} - m\right) r_1\right] A_1 e^{r_1 x} + \left[p - \left(\frac{1}{x} - m\right) r_2\right] A_2 e^{r_2 x} \tag{5.3}$$

$$y' = A_1 r_1 e^{r_1 x} + A_2 r_2 e^{r_2 x} \tag{5.4}$$

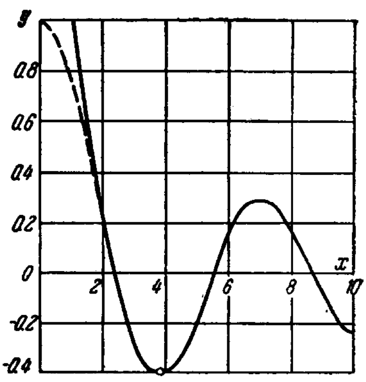
If the initial conditions $x = x_0$, $y = y_0$, $y' = y'= z_0$ are given, then we find A_{10} and A_{20} from (1.10) and then, on the basis of (3.4), we obtain the relation between y and x in the zeroth approximation

$$y = e^{-{}^1/_2 m(x - x_0)} \left[(ab - cd) \cosh k (x - x_0) + \left(\frac{bd - ack^2}{k}\right) \sinh k (x - x_0)\right] \tag{5.5}$$

In the last formula a, b, c, d are determined by means of (3.6). The parameters are found with the aid of (4.4) and (4.5):

$$m = \frac{1}{x - x_0} \int_{x_0}^{x} \frac{dx}{x} = \frac{1}{x - x_0} \ln \frac{x}{x_0}$$

$$p = \frac{1}{x - x_0} \left(\frac{1}{x} - \frac{1}{x_0}\right) + \frac{1}{x - x_0} \int_{x_0}^{x} dx = 1 - \frac{1}{x_0 x} \tag{5.6}$$



In accordance with (5.6) we have

$$e^{-m(x - x_0)} = e^{-\ln (x/x_0)} = x_0 / x$$

Taking into consideration the fact that for large enough x

$$k^2 < 0, \qquad q^2 = p - \frac{m^2}{4} > 0 \tag{5.7}$$

we obtain, by (3.5) and (5.5), the zeroth approximation of the general solution of Equation (5.1)

$$y=\sqrt{\frac{x_0}{x}}\left[(ab-cd)\cos q\,(x-x_0)+\left(\frac{bd+acq^2}{q}\right)\sin q\,(x-x_0)\right] \tag{5.8}$$

As is known, the exact solution of Equation (5.1) is given by Bessel functions in the form

$$y = C_1 I_0(x) + C_2 Y_0(x) \tag{5.9}$$

where C_1 and C_2 are constants, $I_0(x)$ and $Y_0(x)$ are Bessel functions of the zeroth order of the first and second kind.

In the figure, the dotted line represents the exact relation $y = y(x)$ given by (5.9) for $C_2 = 0$ and $C_1 = 1$:

$$y = I_0(x) \tag{5.10}$$

In order to check the accuracy of the new method let the initial conditions be given as $x_0 = 3.88$, $y_0 = -0.40276$, $y_0' = z_0 = 0$ (this point is indicated in the figure) which satisfy the exact solution (5.10), and let us construct the function $y = y(x)$ by means of the formula (5.8) (the solid line in the figure). A comparison of the graphs shows that (5.8) yields practically the same result if $x > 2$. For small x, the error in the computation on the basis of the zeroth approximation increases because of the fact that Equation (5.1) has a singular point when $x = 0$ and $q \to 0$. In particular, we can obtain from (5.8) a simplified approximate formula for large values of x. As x approaches infinity we have

$$m \approx 0, \qquad p \approx 1, \qquad f(x) \approx 0, \qquad q \approx 1, \qquad k^2 = -1$$
$$a \approx y_0 + z_0 f(x_0), \qquad b \approx 1, \qquad c \approx 0, \qquad d \approx z_0$$
$$ab - cd \approx y_0 + z_0 f(x_0) = \text{const}, \qquad \frac{bd + acq^2}{q} \approx z_0 = \text{const}$$

Accordingly, (5.8) yields

$$y=\sqrt{\frac{x_0}{x}}\left[\left(y_0+\frac{z_0}{x_0}\right)\cos(x-x_0)+z_0\sin(x-x_0)\right]=\frac{B_1\cos x}{\sqrt{x}}+\frac{B_2\sin x}{\sqrt{x}}. \tag{5.11}$$

Here B_1 and B_2 are constants. As is known, (5.11) can be obtained with the aid of Bessel functions [3].

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